Multistable collision cycles of Manakov spatial solitons

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We show numerically, using known state-change relations, that collision cycles of Manakov (1+1)-dimensional spatial solitons can exhibit multistable polarization states.

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I. INTRODUCTION

Bistable and multistable optical systems, in addition to being of some theoretical interest, are of practical importance in offering a natural ''flip-flop'' for noise immune storage and logic. We show in this paper that simple cycles of collisions of (1+1)-dimensional spatial solitons governed by the Manakov equations can have more than one distinct stable set of polarization states, and therefore these distinct equilibria can, in theory, be used to store and process information. The multistability occurs in the polarization states of the beams; the solitons themselves do not change shape and remain the usual sech-shaped solutions of the Manakov equations. This is in contrast to multistability in the modes of scalar solitons (see, for example, Ref. [1]). The phenomenon also differs from other examples of polarization multistability in specially engineered devices, such as the vertical-cavity surface-emitting laser (VCSEL) [2], in being dependent only on simple soliton collisions in a completely homogeneous medium.

The picture of *spatial* solitons as self-focused beams is more recent and less well known than the picture of *temporal* solitons propagating along an optical fiber (for example), but the existence and stability of spatial solitons have been well established both theoretically and experimentally in a variety of materials [3]. As pointed out in Ref. [3], bright spatial Kerr solitons are stable only in (1+1)-dimensional systems—that is, systems where the beam can diffract in only one dimension as it propagates. Such solitons are realized in slab waveguides, and are robust with respect to perturbations in both width and intensity. What we show in this paper is that if a realization of vector spatial solitons is governed by the Manakov equations, then multistability is possible in the *steady-state* polarization states of a cycle of beams. The dynamic behavior of such a system in reaching steady-state foci is an open research question, and we leave discussion of this issue for the conclusion of this paper.

The basic configuration considered in this paper requires only that the beams form a closed cycle, and can thus be realized in any nonlinear optical medium that supports spatial Manakov solitons. There are several candidates for physical instantiation of spatial Manakov solitons, including photorefractives [4–8], and semiconductor quantum well wave guides [9]. Very recently, ideal Manakov solitons were also proposed in quadratic media, via optical rectification cascading and the electro-optic effect [10]. For a recent review of optical spatial solitons and their interactions, see Ref. [3].

The work described here is a continuation of ongoing work that aims at exploiting soliton collisions for computation. It was shown in Ref. [11], using explicit solutions of Radhakrishnan *et al.* [12], that collisions of bright Manakov solitons can be described by transformations of a complex-valued polarization state which is the ratio between the two Manakov components. A NOT gate was described there, a MOVE operation was described in Ref. [13], and the general idea of implementing logic using soliton collisions in a homogeneous medium has been studied in Refs. [14–17]. Recently it was shown that allowing time-gating of spatial solitons makes possible the implementation of FANOUT, NAND gates, and hence universal computation [18].

The possibility of using multistable systems of beam collisions broadens the possibilities for practical application of the surprisingly strong interactions that Manakov solitons can exhibit, a phenomenon originally described in Ref. [12]. We show here that a cycle of three collisions can have two distinct foci surrounded by basins of attractions, and that a cycle of four collisions can have 3. Many questions are left for future work: How can one switch effectively and reliably between two foci? Does this phenomenon occur in other vector soliton systems, such as the nonintegrable saturable systems in photorefractives [4–8]? Can such multistable systems be coupled to implement logical operations such as shift registers and arithmetic? We return to these questions in the last section.

II. MATHEMATICAL FRAMEWORK

We review the model and mathematical results we will use. The Manakov system consists of two coupled 3-NLS equations

$$\begin{aligned} i q_{1t} + q_{1xx} + 2\mu(|q_1|^2 + |q_2|^2) q_1 = 0, \\ i q_{2t} + q_{2xx} + 2\mu(|q_1|^2 + |q_2|^2) q_2 = 0, \end{aligned} \tag{1}$$

where $q_1 = q_1(x,t)$ and $q_2 = q_2(x,t)$ are two interacting optical components, μ is a positive parameter, and x and t are normalized space and time. Note that in order for t to represent the propagation variable, as in Manakov's original paper [19], our variables x and t are interchanged with those of Ref. [12]. Furthermore, in our picture of spatial solitons, the variables x and t will represent the horizontal and vertical coordinates of the medium, with t being the direction of beam propagation.

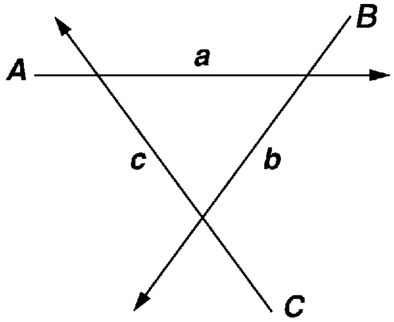

FIG. 1. The basic cycle of three collisions.

The system admits single-soliton, two-component solutions that can be characterized by the complex number $k = k_R + i \cdot k_I$, where k_R determines the energy of the soliton, and k_I is the velocity, all in normalized units, and a complex state ρ, constant between collisions, which is the ratio between the q_1 and q_2 components. The two components can be thought of as components in two directions of polarization, but in the case of a photorefractive crystal are in fact two different uncorrelated beams.

Consider a two-soliton collision, and let k_1 and k_2 represent the constant soliton parameters, associated with the right-moving and left-moving solitons, respectively. Let ρ_1 and ρ_L denote the respective soliton states before impact. Suppose the collision transforms ρ_1 into ρ_R, and ρ_L into ρ_2.

The state change undergone by each colliding soliton takes on the very simple form of a linear fractional transformation (LFT) (also called bilinear or Möbius transformation) [11]. The coefficients are simple functions of the state of the other soliton in the collision. Explicitly, the LFT giving the state of the emerging left-moving soliton is

$$\rho_2 = \frac{[(1-g)/\rho_1^* + \rho_1]\rho_L + g\rho_1/\rho_1^*}{g\rho_L + (1-g)\rho_1 + 1/\rho_1^*}, \tag{2}$$

where

$$g = g(k_1, k_2) = \frac{k_1 + k_1^*}{k_2 + k_1^*} = \frac{2k_{1R}}{k_{1R} + k_{2R} - i\Delta}, \tag{3}$$

and $\Delta = k_{1I} - k_{2I}$, the velocity difference. We assume here, without loss of generality, that $k_{1R}, k_{2R} > 0$. Notice that the transformation from ρ_1 to ρ_2 is not Möbius, but is much more nonlinear. This fact plays a crucial role when we form a closed cycle of collisions, and it appears that it is just this high degree of nonlinearity that makes possible the existence of multiple stable configurations of polarization states.

III. THE BASIC THREE-CYCLE AND COMPUTATIONAL EXPERIMENTS

Figure 1 shows the simplest example of the basic scheme, a cycle of three beams, entering in states *A*, *B*, and *C*, with intermediate beams *a*, *b*, and *c* (see Fig. 1). For convenience, we will refer to the beams themselves, as well as their states, as *A*, *B*, *C*, etc. Suppose we start with beam *C* initially turned off, so that $A = a$. Beam *a* then hits *B*, thereby transforming it to state *b*. If beam *C* in then turned on, it will hit *A*, closing the cycle. Beam *a* is then changed, changing *b*, etc., and the cycle of state changes propagates clockwise. The question we ask is whether this cycle converges, and if so, whether it will converge with any particular choice of complex parameters to exactly zero, one, two, or more foci. We answer the question with numerical simulations of this cycle.

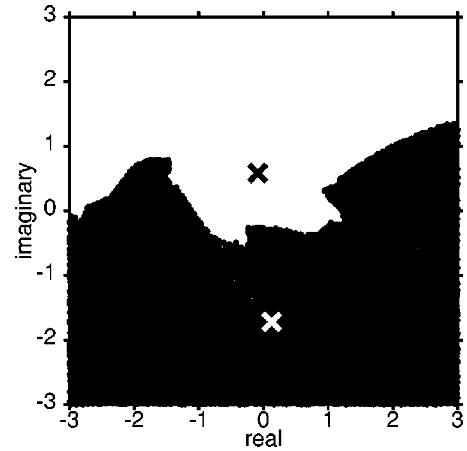

FIG. 2. The two foci and their corresponding basins of attraction in the first example, which uses a cycle of three collisions. The states of the input beams are $A = -0.8 - 0.13i$, $B = 0.4 - 0.13i$, $C = 0.5 + 1.6i$; and $k = 4 \pm i$.

A typical computational experiment was designed by fixing the input beams *A*, *B*, *C*, and the parameters k_1 and k_2, and then choosing points *a* randomly and independently with real and imaginary coordinates uniformly distributed in squares of a given size in the complex plane. The cycle described above was then carried out until convergence in the complex numbers *a*, *b*, and *c* was obtained to within 10^{-12} in norm. Distinct foci of convergence were stored and the initial starting points *a* were categorized by which focus they converged to, thus generating the usual picture of basins of attraction for the parameter *a*. Typically this was done for 50 000 random initial values of *a*, effectively filling in the square, for a variety of parameter choices *A*, *B*, and *C*. The following results were observed.

In cases with one or two clear foci, convergence was obtained in every iteration, almost always within one or two hundred iterations.

Each experiment yielded exactly one or two foci.

The bistable cases (two foci) are somewhat less common than the cases with a unique focus, and are characterized by values of k_R between about 3 and 5 when the velocity difference Δ was fixed at 2.

In the next section we next give two specific examples of bistable parameter choices.

IV. BASINS OF ATTRACTION

Figure 2 shows a bistable example, with the two foci and their corresponding basins of attraction. The parameter *k* is fixed in this and all the examples in this paper at $4 \pm i$ for the right- and left-moving beams of any given collision, respectively. The second example, shown in Fig. 3, shows that the

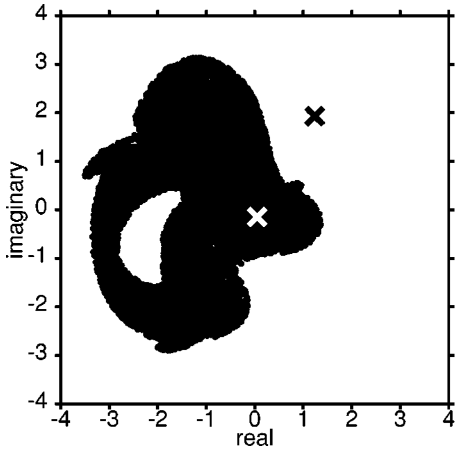

FIG. 3. A second example using a cycle of three collisions, showing that the basins need not be simply connected. The states of the input beams are $A=0.7-0.3i$, $B=-1.1-0.5i$, $C=0.4+0.81i$; and $k=4\pm i$.

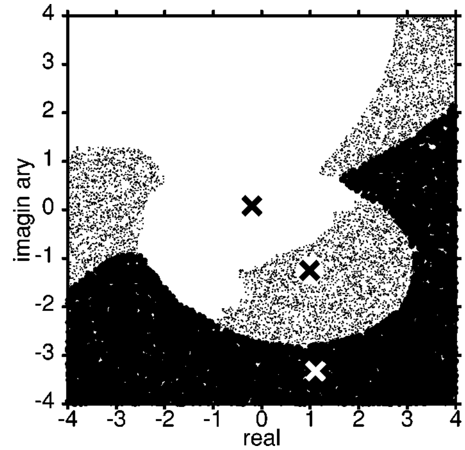

FIG. 5. A case with three stable foci, for a collision cycle of length four. The states of the input beams are $A=-0.39-0.45i$, $B=0.22-0.25i$, $C=0.0+0.25i$, $D=-0.51+0.48i$; and $k=4\pm i$.

basins are not always simply connected; a sizable island that maps to the upper focus appears within the basin of the lower focus.

V. PROPOSED PHYSICAL ARRANGEMENT

Our computations assume that the angles of collisions, which for spatial solitons are determined by the unnormalized velocities in laboratory units, are equal. In situations with strong interactions the angles are small, on the order of a few degrees, at the most. We can arrange that all three collisions take place at the same angle by feeding back one of the beams using mirrors, using an arrangement similar to that shown in Fig. 4. Whether such an arrangement is experimentally practical is left open for future study, but it does not appear to raise insurmountable problems. Note that it is also necessary to divert the continuation of some beams to avoid unwanted collisions.

VI. A TRISTABLE EXAMPLE USING A FOUR-CYCLE

Collision cycles of length four seem to exhibit more complex behavior than those of length 3, although it is difficult to draw any definite conclusions because the parameter spaces are too large to be explored exhaustively, and there is at present no theory to predict such highly nonlinear behavior. If one real degree of freedom is varied as a control parameter, we can move from bistable to tristable solutions, with a regime between in which one basin of attraction disintegrates into many small separated fragments. Clearly, this model is complex enough to exhibit many of the well-known features of nonlinear systems.

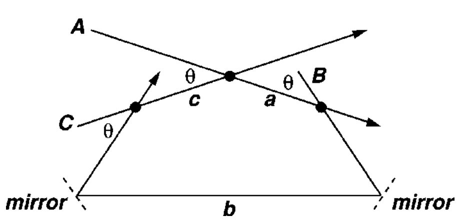

FIG. 4. One way to control the collision angles.

Fortunately, it is not difficult to find choices of parameters that result in very well behaved multistable solutions. For example, Fig. 5 shows such a tristable case. The smallest distance from a focus to a neighboring basin is on the order of 25% of the interfocus distance, indicating that these equilibria will be stable under reasonable noise perturbations.

VII. DISCUSSION

As mentioned in the Introduction, we have exhibited collision configurations of three and four cycles in the Manakov system where there are two or three well separated foci. The general phenomenon raises many questions, both of a theoretical and practical nature. We discuss these in the subsections below.

A. Nonintegrable systems

The fact that there are simple polarization-multistable cycles of collisions in a Manakov system suggests that similar situations occur in other vector systems, particularly photorefractives with a saturable nonlinearity. Any vector system with the possibility of a closed cycle of soliton collisions becomes a candidate for multistability, and there is at this point really no compelling reason to restrict attention to the Manakov case, except for the fact that the explicit state-change relations make numerical study much easier.

B. Dynamics

The simplified picture we used of information traveling clockwise after we begin with a given beam a gives us stable polarization states when it converges, plus an idea of the size of their basins of attractions. It is remarkable that in all cases in our computational experience, except for borderline transitional cases in going from two to three foci in a four cycle, this circular process converges consistently and quickly. But understanding the actual dynamics and convergence characteristics in a real material requires careful physical modeling. This modeling will depend on the nature of the medium used to approximate the Manakov system, and is left for future work. The implementation of a practical way to switch from one stable state to another is likewise critically dependent on the dynamics of soliton formation and perturbation in the particular material at hand, and must be studied with reference to a particular physical realization.

We remark also that no iron-clad conclusions can be drawn from computational experiments about the numbers of foci in any particular case, or the number possible for a given size cycle—despite the fact that we regularly used 50 000 random starting points. On the other hand, the clear cases that have been found, such as those used as examples, are very characteristic of universal behavior in other nonlinear iterated maps, and are sufficient to establish that bistability and tristability, and perhaps higher-mode multistability, is a genuine mathematical characteristic, and possibly also physically realizable. It strongly suggests experimental exploration.

The collision cycles proposed here can also be implemented using collisions of counterpropagating temporal solitons in a fiber. However, this requires a way to divert solitons synchronously after they are used for collisions, and before they interfere with new versions of the fixed solitons used in the cycle. In effect, the periodic bombardment would ''refresh'' the state. This approach, if it could be realized, would directly mimic the iterative algorithm used in this paper to *locate* foci computationally, but as a physical instantiation would be more in the spirit of a dynamic memory than a flip-flop composed of beams in quiescent states.

C. More complicated topologies

We restricted discussion in this paper to the simplest possible structure of a single closed cycle, with three or four collisions. The stable solutions of more complicated configurations are the subject of continuing study. A general theory that predicts this behavior is lacking, and it seems at this point unlikely to be forthcoming. This forces us to rely on numerical studies, from which, as we point out above, only certain kinds of conclusions can be drawn. We are fortunate, however, in being able to find cases that look familiar and which are potentially useful, such as the bistable three-cycles with well separated foci and simply connected basins of attraction.

It is not clear, however, just what algorithms might be used to find equilibria in collision topologies with more than one cycle. It is also intriguing to speculate about how collision configurations with particular characteristics can be designed, how they can be made to interact, and how they might be controlled by pulsed beams. There is promise that when the ramifications of complexes of vector soliton collisions are more fully understood they might be useful for real computation in certain situations. In any event the consequences of the rich interactions possible in collisions of Manakov solitons originally derived by Radhakrishnan *et al.* [12] have yet to be fully explored.

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